6. JORDAN CANONICAL FORM

§6.1. Minimum Polynomials

If you substitute the square matrix A into the characteristic polynomial $\chi_A(\lambda)$ you get the zero matrix. But there may be polynomials of lower degree for which this is true.

A **minimum polynomial** for the square matrix A is a *monic* polynomial $f(\lambda)$ of *lowest degree* with f(A) = 0. Since $\chi_A(x) = 0$, $1 \le \deg f(\lambda) \le n$ for any minimum polynomial $f(\lambda)$ for the $n \times n$ matrix A.

Example 1: The scalar matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ has $\lambda - 2$ as a minimum polynomial.

Example 2: If A = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then (A – I)(A – 2I) = 0, so $(\lambda - 1)(\lambda - 2) = \lambda^2 - 3\lambda + 2$ is a minimum polynomial for A.

Is a minimum polynomial unique? Could we have, for example, a matrix for which both $x^3 + 4x - 2$ and $x^3 - 5x^2 + 7$ are minimum polynomials? The answer is "no", for that would mean that:

$$A^3 + 4A - 2I = 0 \text{ and}$$
$$A^3 - 5A^2 + 7I = 0.$$

Subtracting, we would have $5A^2 + 4A - 9I = 0$, which would contradict the minimality of the degree of the minimum polynomials.

But what if there is no polynomial f(x) for which f(A) = 0? That can't be, as the following theorem shows.

Theorem 1: For any $n \times n$ matrix A over a field F there exists a non-zero polynomial f(x), with coefficients in F, for which f(A) = 0.

Proof: Consider the matrices I, A, A^2, …, A^{n^2}. The space of all $n \times n$ matrices over F has dimension n^2 and here we have $n^2 + 1$ of them. They must therefore be linearly independent over F and so f(A) = 0 for some polynomial of degree at most $n^2 + 1$.

In fact we can do much better than that. We will later see that the minimum polynomial of an $n \times n$ matrix has degree at most n.

Theorem 2: A minimum polynomial for a given matrix is unique.

Proof: Suppose $x^r + a_{r-1}x^{r-1} + \ldots + a_1x + a_0$ and $x^r + b_{r-1}x^{r-1} + \ldots + b_1x + b_0$ be distinct minimum polynomials for A. Then

$$A^r + a_{r-1}A^{r-1} + \ldots + a_1A + a_0I = 0 \text{ and}$$
$$A^r + b_{r-1}A^{r-1} + \ldots + b_1A + b_0I = 0.$$

Subtracting we get:

$(a_{r-1} - b_{r-1})A^{r-1} + \ldots + (a_1 - b_1)A + (a_0 - b_0)I = 0.$

Let $f(x) = (a_{r-1} - b_{r-1})x^{r-1} + \ldots + (a_1 - b_1)x + (a_0 - b_0)$. Since the original two polynomials were distinct this is not the zero poloynomial. It may not have degree $r - 1$ since a_{r-1} and b_{r-1} might be equal, but it certainly is a non-zero polynomial and its degree is less than r. Making it monic we get a contradiction. 🖑☺

Since it is unique we can now refer to it as *the* minimum polynomial and it is now appropriate to give it a notation. We define $\mathbf{m_A(x)}$ to be the minimum polynomial of A. Sometimes we will omit the subscript and just write $\mathbf{m}(x)$.

Theorem 3: For all square matrices A, if f(A) = 0 then $m_A(x)$ divides $f(x)$.

Proof: By the division algorithm for polynomials we may divide f(x) by the minimum polynomial to get a quotient and a remainder.

So $f(x) = m_A(x)q(x) + r(x)$ where either $r(x) = 0$ or

$$\deg r(x) < \deg m_A(x).$$

But $r(A) = f(A) - m_A(A)q(A) = 0 - 0 = 0$. If $r(x) \neq 0$ this would contradict the minimality of the degree of the minimum polynomial. Hence $r(x) = 0$ and so $m_A(x) \mid f(x)$. 🖐☺

Corollary: $m_A(x)$ divides $\chi_A(x)$.

So we can find the minimum polynomial of a matrix by factorising its characteristic polynomial and testing all its divisors.

Example 3: Find the minimum polynomial of

$$A = \begin{pmatrix} 9 & 14 & 21 \\ -7 & -12 & -21 \\ 2 & 4 & 8 \end{pmatrix}.$$

Solution: tr(A) = 5,

$$\mathrm{tr}_2(A) = (-108 + 98) + (72 - 42) + (-96 + 84)$$
$$= -10 + 30 - 12 = 8$$

$$|A| = 9(-96 + 84) - 14(-56 + 42) + 21(-28 + 24)$$
$$= 4.$$

$$\therefore\ \chi(x) = x^3 - 5x^2 + 8x - 4$$
$$= (x-1)(x-2)^2.$$

Hence m(x) divides $(x-1)(x-2)^2$. Clearly it is not a linear factor since A is not a scalar matrix. This leaves three possibilities: $(x-2)^2$, $(x-1)(x-2)$ and $\chi(x)$ itself.

$$(A - 2I)^2 = \begin{pmatrix} 7 & 14 & 21 \\ -7 & -14 & -21 \\ 2 & 4 & 6 \end{pmatrix}\begin{pmatrix} 7 & 14 & 21 \\ -7 & -14 & -21 \\ 2 & 4 & 6 \end{pmatrix}$$
$$= \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & -6 \end{pmatrix} \neq 0.$$

$$(A - I)(A - 2I) = \begin{pmatrix} 8 & 14 & 21 \\ -7 & -13 & -21 \\ 2 & 4 & 7 \end{pmatrix}\begin{pmatrix} 7 & 14 & 21 \\ -7 & -14 & -21 \\ 2 & 4 & 6 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So $m_A(x) = (x-1)(x-2) = x^2 - 3x + 2$.

Once we have proved the next theorem we can reduce the number of factors of $\chi(x)$ that we need to check.

Theorem 4: Every eigenvalue is a zero of the minimum polynomial.

Proof: Let $m(x) = x^r + a_{n-1}x^{r-1} + \ldots + a_1x + a_0$ be the minimum polynomial for A.

Let λ be an eigenvalue of A and let **v** be a corresponding eigenvector.

Then $m(A)\mathbf{v} = (A^r + a_{r-1}A^{r-1} + \ldots + a_1A + a_0I)\mathbf{v}$

$$= (\lambda^r + a_{r-1}\lambda^{r-1} + \ldots + a_1\lambda + a_0)\mathbf{v}$$

But $m(A)\mathbf{v} = 0\mathbf{v} = 0$.

$\therefore$ $(\lambda^r + a_{r-1}\lambda^{r-1} + \ldots + a_1\lambda + a_0)\mathbf{v} = 0$.

Since $\mathbf{v} \neq 0$, $m(\lambda) = \lambda^r + a_{r-1}\lambda^{r-1} + \ldots + a_1\lambda + a_0 = 0$. ✋☺

Corollary: $m_A(x)$ and $\chi_A(x)$ have exactly the same zeros, but with perhaps different multiplicities.

Example 4: Suppose, for some matrix A,

$\chi(x) = (x - 1)^3(x - 2)^2(x - 2)$.

Then we would evaluate the following in turn and we would stop if we got one equal to zero.

$$(A - I)(A - 2I)(A - 3I)$$
$$(A - I)^2(A - 2I)(A - 3I)$$
$$(A - I)(A - 2I)^2(A - 3I)$$
$$(A - I)^2(A - 2I)^2(A - 3I)$$

If none of these are zero then we could conclude that the minimum polynomial is the characteristic polynomial.

As we know, similar matrices have the same trace, determinant and characteristic polynomial. They also have the same minimum polynomial.

Theorem 5: Similar matrices have the same minimum polynomial.

Proof: Suppose that $B = S^{-1}AS$.

Now, for any k, $B^k = (S^{-1}AS)\,(S^{-1}AS) \ldots (S^{-1}AS)$

$$= S^{-1}A^kS.$$

So for any polynomial $f(x)$, $f(B) = S^{-1}f(A)S$ and hence

$$f(B) = 0 \text{ if and only if } f(A) = 0.$$

It follows from Theorem 2 that the minimum polynomials of A and B divide each other, and being monic, they must be equal. ✋☺

Theorem 6: A matrix is diagonalisable if and only if its minimum polynomial has no repeated zeros.

Proof: Let $A = SDS^{-1}$ where $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$ and S is invertible.

Let $\lambda_{u_1}, \ldots, \lambda_{u_r}$ be the distinct eigenvalues.

Then $m_A(x) = m_D(x) = (x - \lambda_{u_1})\,(x - \lambda_{u_2}) \ldots (x - \lambda_{u_r})$.

Conversely suppose that A is an $n \times n$ matrix and suppose that

$m(x) = (x - \lambda_1)(x - \lambda_2) \ldots (x - \lambda_r)$ where the λ_i are distinct.

For each i define $m_i(x) = \dfrac{m(x)}{x - \lambda_i}$, that is m(x) with the factor $x - \lambda_i$ removed.

Then the $m_i(x)$ are coprime and so $1 = m_1(x)k_1(x) + \ldots + m_rk_r(x)$ for some polynomials

$k_1(x), \ldots, k_r(x)$.

Let **v** be any column vector and let $\mathbf{v}_i = m_i(A)k_i(A)\mathbf{v}$ for i = 1, 2, ..., r.

Then $\mathbf{v} = \mathbf{v_1} + \ldots + \mathbf{v_r}$.

Now $(A - \lambda_iI)\mathbf{v_i} = (A - \lambda_i)m_i(A)k_i(A)\mathbf{v}$

$$= m(A)k_i(A)\mathbf{v}$$
$$= 0\ k_i(A)\mathbf{v}$$
$$= \mathbf{0}.$$

Hence each v_i is an eigenvalue, with λ_i being the corresponding eigenvector.

We have therefore shown that every vector is a sum of eigenvectors. The eigenvectors therefore span $\mathbf{R}^n$, and a subset of them will be a basis of eigenvectors. We have shown that this means that A is diagonalisable. ✋☺

We have shown that matrices with distinct eigenvalues and matrices of finite order are diagonalisable. Having proved theorem 3 we can provide much shorter proofs.

Theorem 7: Matrices with no repeated eigenvalues and matrices A such $A^r = I$ for some r, are diagonalisable.

Proof: If $\chi_A(x)$ has no repeated zeros then the same is true of $m_A(x)$.

If $A^r = I$ then $m_A(x)$ divides $x^r - 1$, which has no repeated zeros. ✋☺

§6.2. Jordan Blocks

We've said a lot about diagonalisable matrices, but what about those that have the misfortune to be non-diagonalisable? Can we find something close to a diagonal matrix that they are similar to?

A **Jordan block** is a square matrix of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

There are zeros below the diagonal and the same value down the main diagonal. Every component in the diagonal above the main diagonal has the value 1. Above and to the right of this all the components are zero.

Example 5: A 1×1 Jordan block is just any 1×1 matrix.

A 2×2 Jordan block has the form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

A 3 × 3 Jordan block has the form $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$.

A 4 × 4 Jordan block has the form $\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$.

Jordan blocks of size 2 × 2 and bigger are not diagonalisable. They are the building blocks of non-diagonalisable matrices.

We define the direct sum of square matrices $A_1, A_2, ..., A_r$ to be the matrix

$$A_1 \oplus A_2 \oplus \cdots A_r = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & A_r \end{pmatrix}$$

They are like diagonal matrices except that the A_i's are square matrices and the 0's above are matrices. Diagonal matrices are simply direct sums of 1 × 1 matrices.

Example 6: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \oplus \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \oplus (9) = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 \\ 0 & 0 & 7 & 8 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{pmatrix}$.

Clearly the characteristic polynomial of a direct sum is the product of the corresponding characteristic polynomials.

Example 7: The characteristic polynomial of the above 5 × 5 matrix is
$(\lambda^2 - 5\lambda - 2)(\lambda^2 - 13\lambda - 2)(\lambda - 9)$.

Theorem 8: If A is an n × n matrix and $m_A(x) = (x - \lambda)^n$ then A is similar to a Jordan block.
Proof: Suppose $m_A(x) = (x - \lambda)^n$. Clearly λ is the only eigenvalue.
Since $(A - \lambda I)^{n-1} \neq 0$, the kernel of the linear transformation $v \to (A - \lambda I)^{n-1}v$ is not $\mathbf{R}^n$.
Let **u** be any vector such that $(A - \lambda I)^{n-1}\mathbf{u} \neq \mathbf{0}$.
[If $(A - \lambda I)^{n-1}\mathbf{u} = \mathbf{0}$ for all u then $(A - \lambda I)^{n-1} = 0$, contradicting the minimality of the degree of $\chi_A(x)$.]
For k = 1, 2, 3, ..., n define $\mathbf{v_i} = (A - \lambda I)^{n-k}\mathbf{u}$.
So $(A - \lambda I)\mathbf{v_1} = (A - \lambda I)^n\mathbf{u} = 0$, so $A\mathbf{v_1} = \lambda\mathbf{v_1}$.

If $k \geq 2$ then $(A - \lambda I)\mathbf{v_k} = (A - \lambda I)^{n-(k-1)}\mathbf{u} = \mathbf{v_{k-1}}$ so $A\mathbf{v_k} = \lambda\mathbf{v_k} + \mathbf{v_{k-1}}$.

We now show that $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ is linearly independent. Suppose that $x_1\mathbf{v_1} + ... + x_n\mathbf{v_n} = \mathbf{0}$.
Then $(x_1(A - \lambda I)^{n-1} + x_2(A - \lambda I)^{n-2} + ... + x_{n-1}(A - \lambda I) + x_nI)\mathbf{u} = 0$
Suppose that $x_i \neq 0$ for some m and let m be the largest such subscript.

Then $(x_1(A - \lambda I)^{n-1} + x_2(A - \lambda I)^{n-2} + ... + x_{m-1}(A - \lambda I)^{n-m+1} + x_m(A - \lambda I)^{n-m})\mathbf{u} = 0$
Multiply by $(A - \lambda I)^{m-1}$. Then $x_m(A - \lambda I)^{n-1}\mathbf{u} = 0$.
[All the previous terms disappear since $(A - \lambda I)^n = 0$.]
Since we chose u so that $(A - \lambda I)^{n-1}\mathbf{u} \neq 0$, it follows that $x_m = 0$, a contradiction.

So $x_i = 0$ for all i and hence $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ is linearly independent. This set must therefore be a basis for $\mathbf{R}^n$.

We have $A\mathbf{v_1} = \lambda\mathbf{v_1}$,

$A\mathbf{v_2} = \lambda\mathbf{v}_2 + \mathbf{v_1}$,
$A\mathbf{v_3} = \lambda\mathbf{v_3} + \mathbf{v_2}$,
....
$A\mathbf{v_n} = \lambda\mathbf{v_n} + \mathbf{v_{n-1}}$.

So the matrix of the linear transformation $v \to Av$ relative to the basis $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ is the Jordan block

$$\begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}$$. This means that the matrix A is similar to this Jordan block. ✋☺

§6.3. The Jordan Canonical Form

A Jordan Canonical Form is a direct sum of Jordan blocks. We're going to prove that every square matrix over C is similar to a direct sum of Jordan blocks. Diagonalisable matrices are precisely those where the Jordan blocks are all 1×1 matrices.

Example 8: $$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$ is a direct sum of two Jordan blocks.

Theorem 9: Let M be an $n \times n$ matrix over a field F whose minimum polynomial is a(x)b(x), where a(x) and b(x) are coprime. Then M is similar to $A \oplus B$ for some matrices A, B where $m_A(x) = a(x)$ and $m_B(x) = b(x)$.
Proof: Since a(x), b(x) are coprime, 1 = a(x)h(x) + b(x)k(x) for some a(x), b(x).

Let U be the kernel of the linear transformation **v** $\rightarrow$ a(M)**v** and let V be the kernel of the linear transformation **v** $\rightarrow$ b(M)**v**.

For any vector **v** $\in F^n$, **v** = [a(M)h(M) + b(M)k(M)]**v** so $U \cap V = 0$ and $U + V = F^n$.

Moreover, if **v** $\in$ U then M**v** $\in$ U and if **v** $\in$ V then M**v** $\in$ V.

Let $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_r\}$ be a basis for U and let $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_s\}$ be a basis for V.

Then $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_r, \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_s\}$ is a basis for F^n.

Relative to this basis the matrix for v $\rightarrow$ Mv has the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ where A is an r × r matrix with $m_A(x) = a(x)$ and B is an s × s matrix with $m_B(x) = b(x)$. [Of course n = r + s.] ✋☺

We are now ready to prove the Jordan Canonical form theorem.

Theorem 10: Every square matrix over **C** is similar to a direct sum of Jordan blocks.

Proof: We prove this by induction on the size of the matrix. The theorem is clearly true for

1 × 1 matrices. Suppose that A is an $n \times n$ matrix and that the theorem holds for smaller matrices.

Let the minimum polynomial of A be m(x) = b(x)c(x) where $b(x) = (x - \lambda_1)^r$ and $c(\lambda) \neq 0$.
In other words there are precisely r factors of $x - \lambda_1$ in m(x).]

By theorem 8, A is similar to a direct sum $\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ where B is an $r \times r$ matrix with
$m_B(x) = (x - \lambda_1)^r$ and C is an $s \times s$ matrix with $m_c(x) = c(x)$.
By theorem 7, A is similar to a Jordan block and by induction C is similar to a direct sum of Jordan blocks. It is easy to see that if A_1, A_2 are similar to B_1, B_2 respectively then $A_1 \oplus A_2$ is similar to $B_1 \oplus B_2$. ✋☺

Example 9: If A = $\begin{pmatrix} 3 & -22 & 18 \\ 3 & -14 & 9 \\ 2 & -8 & 4 \end{pmatrix}$ find an invertible matrix S such that $A = SJS^{-1}$ where J is a direct sum of Jordan blocks.

Solution: We obtain the Jordan Canonical Form by following the steps in theorems 7 and 8.
We begin by working out the characteristic polynomial.
tr(A) = 3 – 14 + 4 = –7.
$tr_2(A) = \begin{vmatrix} 3 & -22 \\ 3 & -14 \end{vmatrix} + \begin{vmatrix} 3 & 18 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} -14 & 9 \\ -8 & 4 \end{vmatrix} = -42 + 66 + 12 - 36 - 56 + 72 = 16.$

$$|A| = \begin{vmatrix} 3 & -22 & 18 \\ 3 & -14 & 9 \\ 2 & -8 & 4 \end{vmatrix} = \begin{vmatrix} 3 & -22 & 18 \\ 0 & 8 & -9 \\ 2 & -8 & 4 \end{vmatrix} = 3(32 - 72) + 2(198 - 144) = -120 + 108 = -12.$$

Hence $\chi_A(x) = x^3 + 7x^2 + 16x + 12 = (x + 2)^2(x + 3)$.

$$A + 2I = \begin{pmatrix} 5 & -22 & 18 \\ 3 & -12 & 9 \\ 2 & -8 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -4 & 3 \\ 2 & -8 & 6 \\ 5 & -22 & 18 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -4 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -4 & 3 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\therefore \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$ is an eigenvector for $\lambda = -2$.

$$A + 3I = \begin{pmatrix} 6 & -22 & 18 \\ 3 & -11 & 9 \\ 2 & -8 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3 & 2 \\ 2 & -8 & 7 \\ 6 & -22 & 18 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3 & 2 \\ 0 & -2 & 3 \\ 0 & -4 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\therefore \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$ is an eigenvector for λ = –3.

$$(A + 2I)(A + 3I) = \begin{pmatrix} 5 & -22 & 18 \\ 3 & -12 & 9 \\ 2 & -8 & 6 \end{pmatrix}\begin{pmatrix} 6 & -22 & 18 \\ 3 & -11 & 9 \\ 2 & -8 & 7 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & -12 & * \\ * & * & * \\ * & * & * \end{pmatrix} \neq 0.$$

$\therefore m_A(x) = (x + 2)^2(x + 3)$.

At this stage we could decide that the Jordan Canonical Form must be $J = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ but let us derive it using the proofs of theorems 7 and 8. This would allow us to find an appropriate invertible matrix S (which by the way, is not unique).

Let $a(x) = (x + 2)^2$ and $b(x) = x + 3$.

$$(A + 2I)^2 = \begin{pmatrix} 5 & -22 & 18 \\ 3 & -12 & 9 \\ 2 & -8 & 6 \end{pmatrix}\begin{pmatrix} 5 & -22 & 18 \\ 3 & -12 & 9 \\ 2 & -8 & 6 \end{pmatrix} = \begin{pmatrix} -5 & 10 & 0 \\ -3 & 6 & 0 \\ -2 & 4 & 0 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So $\mathbf{u_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{u_2} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ is a basis for U, the kernel of $\mathbf{v} \to (A + 2I)^2\mathbf{v}$.

Also $\mathbf{v_1} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$ is a basis for V, the kernel of $\mathbf{v} \to (A + 3I)\mathbf{v}$.

Then $\mathbf{u_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{u_2} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v_1} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$ is a basis for $\mathbf{R}^3$.

$A\mathbf{u_1} = \begin{pmatrix} 18 \\ 9 \\ 4 \end{pmatrix} = 4\mathbf{u_1} + 9\mathbf{u_2}$, $A\mathbf{u_2} = \begin{pmatrix} -16 \\ -8 \\ -4 \end{pmatrix} = -4\mathbf{u_1} - 8\mathbf{u_2}$ and

$A\mathbf{v_1} = -3\mathbf{v_1}$.

Relative to this basis the matrix of the linear transformation $\mathbf{v} \to A\mathbf{v}$ is

$\begin{pmatrix} 4 & -4 & 0 \\ 9 & -8 & 0 \\ 0 & 0 & -3 \end{pmatrix} = B \oplus C$ where $B = \begin{pmatrix} 4 & -4 \\ 9 & -8 \end{pmatrix}$ and $C = (-3)$.

The minimum polynomial of B is $(x + 2)^2$. Now $B + 2I = \begin{pmatrix} 6 & -4 \\ 9 & -6 \end{pmatrix} \to \begin{pmatrix} 3 & -2 \\ 0 & 0 \end{pmatrix}$.

Let $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which is a convenient vector outside of the null-space of B + 2I.

Following theorem 7 we take $\mathbf{v_1} = (B + 2I)\mathbf{u} = \begin{pmatrix} 6 & -4 \\ 9 & -6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$ and $\mathbf{v_2} = \mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Then $B\mathbf{v_1} = \begin{pmatrix} 4 & -4 \\ 9 & -8 \end{pmatrix} \begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} -12 \\ -18 \end{pmatrix} = -2\mathbf{v_1}$ and $B\mathbf{v_2} = \begin{pmatrix} 4 & -4 \\ 9 & -8 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix} = \mathbf{v_1} - 2\mathbf{v_2}$.

The matrix of B relative to this basis is $\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$, giving the Jordan form of A as

$J = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$. We would have to do a bit more work to follow through the change of bases to find the appropriate S.

A quicker way to find an appropriate invertible matrix S, once we have determined the J, is to work from first principles. Write S in terms of its columns, as $S = (\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$.

Then AS = (A$\mathbf{v_1}$, A$\mathbf{v_2}$, A$\mathbf{v_3}$) and JS = $\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$($\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$)

= ($-2\mathbf{v_2}$, $\mathbf{v_1} - 2\mathbf{v_2}$, $-3\mathbf{v_3}$)

So we want A$\mathbf{v_1} = -2\mathbf{v_1}$, A$\mathbf{v_2} = \mathbf{v_1} - 2\mathbf{v_2}$ and $\mathbf{v_3} = -3\mathbf{v_3}$.

Take $\mathbf{v_1} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$, $\mathbf{v_3} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$. For $\mathbf{v_2}$ we must solve the system (A + 2I)$\mathbf{v_2} = \mathbf{v_1}$.

$$\left(\begin{array}{ccc|c} 5 & -22 & 18 & 6 \\ 3 & -12 & 9 & 3 \\ 2 & -8 & 6 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 5 & -22 & 18 & 6 \\ 1 & -4 & 3 & 1 \\ 2 & -8 & 6 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -4 & 3 & 1 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \to$$

$\left(\begin{array}{ccc|c} 1 & -4 & 3 & 1 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$. Take $\mathbf{v_2} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

Hence if S = $\begin{pmatrix} 6 & 2 & 5 \\ 3 & 1 & 3 \\ 2 & 1 & 2 \end{pmatrix}$ we have A = SJS^{-1}.

§6.4. Powers of Jordan Blocks

The Binomial Coefficients $\binom{m}{r}$ are normally only defined if $0 \le r \le m$. However we'll allow r to go outside this range by defining $\binom{m}{r} = 0$ when r < 0 or $r > m$.

Theorem:11: If J is an $n \times n$ Jordan Block,

$$\begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix} \text{ then } J^m =$$

$$\begin{pmatrix} \lambda^m & \binom{m}{1}\lambda^{m-1} & \binom{m}{2}\lambda^{m-2} & \dots & \binom{m}{n-2}\lambda^{m-n+2} & \binom{m}{n-1}\lambda^{m-n+1} \\ 0 & \lambda^m & \binom{m}{1}\lambda^{m-1} & \dots & \binom{m}{n-3}\lambda^{m-n+3} & \binom{m}{n-2}\lambda^{m-n+2} \\ 0 & 0 & \lambda^m & \dots & \binom{m}{n-4}\lambda^{m-n+4} & \binom{m}{n-3}\lambda^{m-n+3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda^m & \binom{m}{1}\lambda^{m-1} \\ 0 & 0 & 0 & \dots & 0 & \lambda^m \end{pmatrix}$$

The components on each diagonal are the constant. Numbering the main diagonal as diagonal 0, the diagonals above as 1, 2, 3 … and the diagonals below as –1, –2, –3, …, the components on the r'th diagonal are all equal to

$\binom{m}{r}\lambda^{m-r}$. So the components below the main diagonal are zero.

Proof: Let $J^m = (a_{ij}^{(m)})$.

I will prove by induction on m that $a_{ij}^{(m)} = \binom{m}{i-j}\lambda^{m-i+j}$.

This uses the identity $\binom{m}{r} = \binom{m-1}{r-1} + \binom{m-1}{r}$. ✋☺

Example 10: If $J = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$ then $J^m = \begin{pmatrix}\lambda^m & m\lambda^{m-1}\\ 0 & \lambda^m\end{pmatrix}$.

If $J = \begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{pmatrix}$ then $J^m = \begin{pmatrix}\lambda^m & m\lambda^{m-1} & \binom{m}{2}\lambda^{m-2}\\ 0 & \lambda^m & m\lambda^{m-1}\\ 0 & 0 & \lambda^m\end{pmatrix}$.

If J is a direct sum of Jordan blocks we raise each of the blocks independently.

Example 11: If $J = \begin{pmatrix}J_1 & 0\\ 0 & J_2\end{pmatrix}$ where J_1 and J_2 are Jordan blocks then $J^m = \begin{pmatrix}J_1^m & 0\\ 0 & J_2^m\end{pmatrix}$.

§6.5. Will That Asteroid Ever Hit The Earth?

Imagine that we have the equation of motion of an asteroid. We could potentially use it to work out whether it will ever hit the earth. The problem we're discussing next is a sort of discrete version of this scenario. We suppose that we know the current position of an object and we have a function that gives its position after 1 unit of time. Will this moving object ever reach a certain location?

Of course there's the difficulty that the earth and the asteroid are not points in space. If we represented their location by their centres of gravity we'd get a disastrous situation if they even came within thousands of kilometres of each other. Here we're talking about *exact* positions.

Because were discussing linear algebra we'll assume that the function that takes an object at position **v** to its position, one unit of time later, is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ (no need to stick to only 3 dimensions). So it can be represented by an $n \times n$ matrix.

If the movement in one unit of time is $\mathbf{v} \to A\mathbf{v}$, then after m units of time we'll have $\mathbf{v} \to A^{m}\mathbf{v}$. So if our 'target' is at **c**, we want to know whether an object currently at position **v**, will ever hit the target at position **c**. In other words we want to know if the equation $A^m\mathbf{v} = \mathbf{c}$ has a solution for m, given **v** and **c**.

We find a direct sum of Jordan blocks, J, that is similar to the matrix A. This, we can always do. Then, if $A = SJS^{-1}$ then $A^m\mathbf{v} = SJ^mS^{-1}\mathbf{v}$. Hence we can decide whether there is a solution.

Example: Let $A = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 1001 \\ -498 \end{pmatrix}$. Will $A^m\mathbf{v} = \mathbf{c}$ for any m?

Solution: $\chi_A(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$ so there is a repeated eigenvalue of 1.

$A - I = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}$ so $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is the 'only' eigenvector. In other words the eigenspace is only 1-dimensional.

The Jordan Canonical Form is clearly $J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let S = (**u**, **w**) so that AS = SJ.

Then A(**u**, **w**) = (**u**, **w**) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and hence A**u** = **u** and A**w** = **u** + **w**.

Take $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. We must find **w** such that (A – I)**w** = **u**.

$A - I = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}$ so we adjoin u to get $\left(\begin{array}{cc|c} 2 & 4 & 2 \\ -1 & -2 & -1 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right)$. So take $\mathbf{w} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Hence $S = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$. Now $S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

Hence $A = SJS^{-1}$ and so $A^m = SJ^mS^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

Therefore $A^m\mathbf{v} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5m + 3 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 10m + 1 \\ -5m + 2 \end{pmatrix}.$$

So if $A^m\mathbf{v} = \begin{pmatrix} 1001 \\ -498 \end{pmatrix}$ we must have $\begin{cases} 10m + 1 = 1001 \\ -5m + 2 = -498 \end{cases}$.

These equations are consistent, so there is a solution, namely $m = 100$.

If this was an asteroid potentially hitting a 2-dimensional earth, and the unit of time was years, the asteroid would hit the earth after 100 years.

If $\mathbf{c} = \begin{pmatrix} 1000 \\ -500 \end{pmatrix}$ the asteroid would never hit the earth because the system of equations $\begin{cases} 10m + 1 = 1000 \\ -5m + 2 = -500 \end{cases}$ is inconsistent. However, depending on the units of distance, it could be a close thing in 100 years!

If $\mathbf{c} = \begin{pmatrix} 1005 \\ -500 \end{pmatrix}$ the system $\begin{cases} 10m + 1 = 1005 \\ -5m + 2 = -500 \end{cases}$ has a solution $m = 100.4$. So the asteroid would hit the earth in just over 100 years.

It should be pointed out that the 'application' to asteroids hitting the earth is purely a way of depicting the mathematical problem and it is not a realistic astronomical model. There are applications of this technique, but they are not nearly so dramatic!

EXERCISES FOR CHAPTER 6

Exercise 1: (a) Write down all 5×5 direct sums of Jordan blocks J whose spectrum is $\{5\}$.
(b) For each of the above matrices find its minimum polynomial.
(c) Which of the above matrices are similar?

Exercise 2: If $A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \\ -1 & 0 & 4 \end{pmatrix}$ find an invertible matrix S such that $A = SJS^{-1}$ where J is a direct sum of Jordan blocks.

Exercise 3: Explain why $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ is diagonalisable.

Exercise 4: Suppose A is a non-diagonalisable matrix such that $A^k = \lambda I$ for some k and $\lambda \geq 0$. Find λ.

Exercise 5: Prove that for a square matrix A the number of Jordan blocks in its Jordan Canonical Form is the dimension of the eigenspace E_A.

SOLUTIONS FOR CHAPTER 6

Exercise 1: (a)

$$J_1 = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

$$J_2 = \begin{pmatrix} 5 & 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

$$J_4 = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

$$J_5 = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

$$J_6 = \begin{pmatrix} 5 & 1 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

$$J_7 = \begin{pmatrix} 5 & 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

$$J_8 = \begin{pmatrix} 5 & 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

$$J_9 = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

$$J_{10} = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

$$J_{11} = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

$$J_{12} = \begin{pmatrix} 5 & 1 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

$$J_{13} = \begin{pmatrix} 5 & 1 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

$$J_{14} = \begin{pmatrix} 5 & 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

$$J_{15} = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix},$$

$$J_{16} = \begin{pmatrix} 5 & 1 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

minimum polynomial	**examples**
$(x - 5)^5$	J_{16}
$(x - 5)^4$	J_{12}, J_{15}
$(x - 5)^3$	J_6, J_9, J_{11}, J_{13}, J_{14}
$(x - 5)^2$	J_2, J_3, J_4, J_5, J_7, J_8, J_{10}
$x - 5$	J_1

Similarity classes:

$\mathbf{J_1}$	$\mathbf{J_2, J_3, J_4, J_5}$	$\mathbf{J_6, J_9, J_{11}}$
1 + 1 + 1 + 1 + 1	2 + 1 + 1 + 1	3 + 1 + 1

$\mathbf{J_7, J_8, J_{10}}$	$\mathbf{J_{12}, J_{15}}$	$\mathbf{J_{13}, J_{14}}$	$\mathbf{J_{16}}$
2 + 2 + 1	4 + 1	3 + 2	5

The second row gives the sizes of the Jordan blocks.

Exercise 2: $\text{tr}(A) = 9$.

$$\text{tr}_2(A) = \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 0 & 4 \end{vmatrix} = 7 + 8 + 12 = 27.$$

$$|A| = \begin{vmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \\ -1 & 0 & 4 \end{vmatrix} = 2(12) - (-3) = 27.$$

$\chi_A(x) = x^3 - 9x^2 + 27x - 27 = (x - 3)^3$.

So 3 is the only eigenvalue.

$A - 3I = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ so $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ spans the eigenspace $E_A(3)$.Clearly A is not diagonalisable. Moreover if the Jordan Canonical form is $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ the dimension of $E_A(3)$ would be 2. So A is similar to $J = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

Let $S = (\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$ be an invertible matrix.

Then $AS = (A\mathbf{v_1}, A\mathbf{v_2}, A\mathbf{v_3})$ and $SJ = (3\mathbf{v_1}, \mathbf{v_1} + 3\mathbf{v_2}, \mathbf{v_2} + 3\mathbf{v_3})$, so to get $AS = SJ$ we need

$A\mathbf{v_1} = 3\mathbf{v_3}$, $A\mathbf{v_2} = \mathbf{v_1} + 3\mathbf{v_2}$, $A\mathbf{v_3} = \mathbf{v_2} + 3\mathbf{v_3}$.

Take $\mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$ so take $\mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$ so

take $\mathbf{v_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Then if $S = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, $A = SJS^{-1}$.

Exercise 3: Although A looks like a Jordan block it has distinct eigenvalues 1, 2, 3 and so it really is diagonalisable. Remember that the diagonal components of a Jordan block have to be equal.

Exercise 4: If $\lambda \neq 0$ and $B = \frac{1}{\lambda^{1/k}} A$ then $B^k = \frac{1}{\lambda} A^k = I$. Since matrices of finite order are diagonalisable B, and hence A, is diagonalisable. So $\lambda = 0$.

Exercise 5: The eigenspace of a direct sum of Jordan blocks is the direct sum of the eigenspaces of the individual Jordan blocks. The dimension of the eigenspace of a Jordan block $B = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$ is 1 since $B - \lambda I = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and the only solutions of the equation $(B - \lambda I)\mathbf{v} = \mathbf{0}$ are the scalar multiples of $\begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}$.

FURTHER EXERCISES

If you need further practice with Jordan Canonical Forms why not make up your own examples by working backwards. Choose a Jordan Canonical Form J and an invertible matrix T and compute $A = TJT^{-1}$. Now work out the eigenvalues and eigenvectors, the Jordan Canonical Form and a suitable matrix S such that $A = SJS^{-1}$. Your J should be the Jordan Canonical Form you started with (perhaps with the Jordan blocks rearranged. But note that your S need not be the same as the T you began with. Check your answer by computing SJS^{-1}.

To make the arithmetic pleasant you should keep your eigenvalues between –9 and 9 and your invertible matrix should have determinant 1 or –1.

Here are some T's that have this property.

$$\begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 2 \\ 5 & 9 & 11 & 5 \\ 3 & 4 & 8 & 4 \end{pmatrix}.$$